Homology in an Abelian Category

Joshua Ruiter

October 15, 2019

In many algebra textbooks, after defining the category of chain complexes of R-modules and associated homology functors, the author notes that the same construction holds when replacing R-mod with a general abelian category. However, rarely does the author carry out this construction in that general setting. The purpose of these notes is to carry out that construction in as much detail as possible.

Contents

1 Abelian Categories

We assume the reader is familiar with the categorical notions (and associated universal properties) of kernels, cokernels, monomorphisms, epimorphisms, zero objects, products, and functors. We also assume familiarity with abelian groups and the category of left modules over a commutative ring R with unity (this category is denoted R-mod). In these notes, a ring R will always be a commutative ring with unity.

In many ways, the notion of an abelian category is an attempt to describe what makes R-mod such a good category in purely categorical terms, without reference to sets and elements.

Definition 1.1. *An* ***abelian category*** *is a category $\mathcal{A}$ in which*

1. *For any two objects X, Y, $\mathrm{Hom}(X, Y)$ is an abelian group; in particular, there is a zero morphism. Composition of morphisms distributes over addition,*

$$f(g+h) = fg + fh \qquad (f+g)h = fh + gh$$

assuming all of the morphisms have the appropriate domain and codomain.

2. *Every morphism $f : X \to Y$ has a kernel and a cokernel.*
3. *There is a zero object.*
4. *For every pair of objects X, Y, the product $X \times Y$ exists.*
5. *Every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.*

Examples: The category of abelian groups forms an abelian category. The category R-mod forms an abelian category. These are the primary and motivational examples of abelian categories.

What follows is a list of rather dry but important properties of abelian categories. Note that each of these lemmas is used in our construction of the homology functor.

Lemma 1.1. *Let $\mathcal{A}$ be an abelian category and $f : X \to Y$ be a morphism. Then f is epi if and only if for every $g : Y \to Z$ so that $gf = 0$, we have $g = 0$. Dually, f is mono if and only if for every $h : W \to X$ such that $fh = 0$, we have $h = 0$.*

Proof. Suppose f is mono, and $fh = 0$. We know that $f \circ 0 = 0$, so by definition of monomorphism, $h = 0$. Now suppose that $fh = 0$ implies $h = 0$. Then if we have $e_1, e_2 : W \to X$ such that $fe_1 = fe_2$, we get $f(e_1 - e_2) = 0$ so $e_1 = e_2$, and thus f is mono.

The statement involving epimorphisms is proved by the same argument with compositions in the other order. □

Lemma 1.2. *Let $\mathcal{A}$ be an abelian category and $f : X \to Y$ a morphism with kernel $k : K \to X$ and cokernel $c : Y \to C$. Then k is mono and c is epi.*

Proof. First we show that k is mono. Let $g_1, g_2 : Z \to K$ so that $k \circ g_1 = k \circ g_2 : Z \to X$. We need to show that $g_1 = g_2$. Since $f \circ k = 0$, we have $f \circ (k \circ g_1) = 0$, so by the universal property of the kernel, there is a unique morphism $Z \to K$ making the following diagram commute.

$$\begin{tikzcd}
 & X \arrow[dr, "f"] & \\
Z \arrow[ur, "k \circ g_1"] \arrow[r, dotted, "\phi"'] & K \arrow[u, "k"'] \arrow[r, "0_{KY}"'] & Y
\end{tikzcd}$$

We know that g_1 makes it commute, so $\phi = g_1$. Since $k \circ g_2$, g_2 also makes this commute, so $\phi = g_2$. Thus $g_1 = g_2$, so k is mono.

To show that c is epi, we basically do the same argument with arrows reversed. Let $h_1, h_2 : C \to Z$ be morphisms so that $h_1 \circ c = h_2 \circ c$. We need to show that $h_1 = h_2$. By the universal property of the cokernel, there is a unique morphism $\psi : C \to Z$ so that the following diagram commutes.

$$\begin{tikzcd}
 & Y \arrow[d, "c"] \arrow[dr, "h_1 \circ c"] & \\
X \arrow[ur, "f"] \arrow[r, "0_{XC}"'] & C \arrow[r, dotted, "\psi"'] & Z
\end{tikzcd}$$

Then $\psi = h_1$ by uniqueness, and since $h_1 \circ c = h_2 \circ c$, $h_2 = \psi$ again by uniqueness, so $h_1 = h_2$. Thus c is epi. $\square$

Lemma 1.3. *Let $f : X \to Y$ and $g : Y \to Z$. If f, g are both epimorphisms, then $gf : X \to Z$ is an epimorphism. If f, g are both monomorphisms, then $gf : X \to Z$ is a monomorphism.*

Proof. First we prove the statement about epimorphisms. Suppose we have maps $h_1, h_2 : Z \to W$ such that $h_1 gf = h_2 gf$.

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & W
\end{tikzcd}$$

Since f is an epimorphism $h_1 g = h_2 g$. Then since g is epi, $h_1 = h_2$. Thus gf is epi. Now for the statement about monomorphisms. Suppose we have maps $m_1, m_2 : M \to X$ so that $gfm_1 = gfm_2$. Then since g is mono, $fm_1 = fm_2$, then since f is mono, $m_1 = m_2$, and hence gf is mono. $\square$

Lemma 1.4. *Let $f : X \to Y$ and $g : Y \to X$ be morphisms so that $gf = \mathrm{Id}_X$. Then g is epi and f is mono.*

Proof. Suppose we have $h_1, h_2 : X \to Z$ so that $h_1 g = h_2 g$. Then $h_1 gf = h_2 gf \implies h_1 = h_2$, so g is epi. Similarly, if we have $m_1, m_2 : M \to X$ so that $fm_1 = fm_2$, then $gfm_1 = gfm_2 \implies m_1 = m_2$, so f is mono.

$$\begin{tikzcd}
M \arrow[r, shift left, "m_1"] \arrow[r, shift right, "m_2"'] & X \arrow[r, "f"] & Y \arrow[r, "g"] & X \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & Z
\end{tikzcd}$$

$\square$

Definition 1.2. *Let $\mathcal{A}$ be an abelian category and $f : X \to Y$ a morphism. The* ***image*** *of f is the* $\ker(\operatorname{coker}(f))$*. We can depict this as below, where* $(\operatorname{coker} f, q)$ *is the cokernel of f, and* $(\operatorname{im} f, i)$ *is the kernel of q.*

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "q"] & \operatorname{coker} f \\
& \operatorname{im} f \arrow[u, "i"] &
\end{tikzcd}$$

Example: In the category of R-modules, the cokernel of $f : X \to Y$ is the projection $\pi : Y \to Y/\operatorname{im} f$, where $\operatorname{im} f$ refers to the usual set-theoretic image of f. The kernel of π is the injection $\operatorname{im} f \hookrightarrow Y$. Thus the set-theoretic image of f agrees with the category-theoretic image.

Proposition 1.5. *Let $f : A \to B$ be a morphism. There is a unique morphism $\widetilde{f} : A \to \operatorname{im} f$ making the triangle below commute, and $\widetilde{f}$ is an epimorphism.*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\widetilde{f}"'] & B \\
\operatorname{im} f \arrow[ur, "i"'] &
\end{tikzcd}$$

(The map i is the kernel of the cokernel of f.)

Proof. Note that the existence and uniqueness of $\widetilde{f}$ are not the hard part of this; the hard part is showing that $\widetilde{f}$ is epi. To start, here is a roadmap, a diagram that includes all of the maps involved in the proof.

$$\begin{tikzcd}
& & & \operatorname{coker} is \\
A \arrow[rr, "f"] \arrow[dr, "\widetilde{f}"] \arrow[dd, "t"] & & B \arrow[ur, "r"] \arrow[dr, "\ell"] & \\
& \operatorname{im} f \arrow[ur, "i"] \arrow[dr, "y"] \arrow[dl, bend left, "\psi"] & & \operatorname{coker} f \arrow[uu, "\phi"'] \\
\ker y \arrow[ur, "s"] & & Y &
\end{tikzcd}$$

Let $\ell : B \to \operatorname{coker} f$ be the cokernel of f, and let $\ell : B \to \operatorname{coker} f$ be the kernel of ℓ, which is also the image of f. Note that $\ell f = 0$ by definition of cokernel. First, we construct $\widetilde{f}$. By the universal property of $\operatorname{im} f$ being the kernel of ℓ, there is a unique map $\widetilde{f} : A \to \operatorname{im} f$ making the following diagram commutes.

$$
\begin{tikzcd}
& B \arrow[dr, "\ell"] & \\
& \operatorname{im} f \arrow[u, "i"] \arrow[r, "0"] & \operatorname{coker} f \\
A \arrow[uur, bend left, "f"] \arrow[ur, dotted, "\widetilde{f}"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
$$

This establishes the existence and uniqueness of $\widetilde{f}$, now we just need to show that it is epi. By Lemma 1.1, we can show that f is epi by taking an arbitrary morphism $y : \operatorname{im} f \to Y$ such that $y\widetilde{f} = 0$ and showing that $y = 0$, so this is what we will do. Let $y : \operatorname{im} f \to Y$ be a morphism satisfying $y\widetilde{f} = 0$. Let $s : \ker y \to Y$ be the kernel of y. By the universal property of the kernel, there is a unique $t : A \to \ker y$ making the following diagram commute. (Note that $ys = 0$ because s is the kernel of y.)

$$
\begin{tikzcd}
& \operatorname{im} f \arrow[dr, "y"] & \\
& \ker y \arrow[u, "s"] \arrow[r, "0"] & \operatorname{coker} f \\
A \arrow[uur, bend left, "\widetilde{f}"] \arrow[ur, dotted, "t"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
$$

Note that s and i are mono by Lemma 1.2, so the composition is is mono by Lemma 1.3. Let $r : B \to \operatorname{coker} is$ be the cokernel of is. Note that $ris = 0$. Now observe that

$$rf = ri\widetilde{f} = rist = 0t = 0$$

so by the universal property of the cokernel $\operatorname{coker} f$, there is a unique $\phi : \operatorname{coker} f \to \operatorname{coker} is$ making the following diagram commute.

$$
\begin{tikzcd}
& B \arrow[d, "\ell"] \arrow[ddr, bend left, "r"] & \\
A \arrow[ur, "f"] \arrow[r, "0"] \arrow[drr, bend right, "0"'] & \operatorname{coker} f \arrow[dr, dotted, "\phi"] & \\
& & \operatorname{coker} is
\end{tikzcd}
$$

Thus $ri = \phi\ell i = \phi 0 = 0$. Since $\mathcal{A}$ is an abelian category, is is the kernel of its cokernel r. By the universal property of is being the kernel of r, there is a unique $\psi : \operatorname{im} f \to \ker y$ making the following diagram commute.

$$
\begin{tikzcd}
& B \arrow[dr, "r"] & \\
& \ker y \arrow[u, "is"] \arrow[r, "0"] & \operatorname{coker} is \\
\operatorname{im} f \arrow[uur, bend left, "i"] \arrow[ur, dotted, "\psi"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
$$

We then have $is\psi = i$, which implies that $s\psi = \mathrm{Id}$ since i is mono. This implies that s is epi (using Lemma 1.4). Then since $ys = 0$, we get $y = 0$ by Lemma 1.1. As noted above, since y was an arbitrary morphism such that $y\widetilde{f} = 0$, this implies that $\widetilde{f}$ is epi by Lemma 1.1. □

2 Category of Chain Complexes

We assume the reader has already encountered chain complexes of abelian groups or of R-modules, perhaps in the context of algebraic topology. Chain complexes of abelian groups are a very useful tool in algebrac topology to give computable algebraic invariants to distinguish spaces like the sphere and the torus. The point of this section is that chain complexes still make sense in this generalized setting of an arbitrary abelian category, and that chain complexes then form another abelian category.

Definition 2.1. *Let $\mathcal{A}$ be an abelian category. A* ***chain complex*** *C in $\mathcal{A}$ is a family of objects $\{C_n\}_{n\in\mathbb{Z}}$ along with a family of morphisms $d_n : C_n \to C_{n-1}$ so that $d_n \circ d_{n+1} = 0$ for all n.*

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \longrightarrow \cdots$$

The maps d_n are called ***differentials****. A* ***chain map*** *between chain complexes B_*, C_* is a sequence of morphisms $f_n : B_n \to C_n$ so that the following diagram commutes.*

$$\begin{tikzcd}
\cdots \arrow[r] & B_{n+1} \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & B_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & B_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

We compose morphisms $f = (f_n), g = (g_n)$ by $g \circ f = (g_n \circ f_n)$. Now we can form the category $\mathrm{Kom}(\mathcal{A})$ *with objects as chain complexes in $\mathcal{A}$ and morphisms as chain maps.*

Proposition 2.1. *Let $\mathcal{A}$ be an abelian category. Then* $\mathrm{Kom}(\mathcal{A})$ *is an abelian category.*

Proof. We need to check the following things:

1. Composition is associative.
2. There are identity arrows.
3. $\mathrm{Hom}(X, Y)$ is an abelian group.
4. Composition is bilinear with respect to the group structure.
5. There is a zero object.
6. Pairwise products exist.
7. Kernels and cokernels exist.
8. Every monic is the kernel of its cokernel and every epi is the cokernel of its kernel.

(1) Associativity of composition follows from associativity of composition in $\mathcal{A}$.
(2) The identity chain map on (C_n, d_n) is (Id_{C_n}).
(3) Let B, C be chain complexes. If $f = (f_n), g = (g_n) \in \mathrm{Hom}(C, D)$ then we define $f + g = (f_n + g_n)$. Then $\mathrm{Hom}(B, C) = \prod_{n\in\mathbb{Z}} \mathrm{Hom}(C_n, D_n)$, which is an abelian group.
(4) Linearity follows from linearity in $\mathcal{A}$. For example,

$$(f+g)\circ h = ((f_n + g_n)\circ h_n) = (f_n \circ h_n + g_n \circ h_n) = (f_n \circ h_n) + (g_n \circ h_n) = f\circ h + g \circ h$$

(5) The zero object is the chain complex with all objects zero and all morphisms zero. For any chain complex $C = (C_n, d_n)$, there is exactly one chain map $0 \to C$, which has all morphisms zero. Similarly, it is terminal.
(6) Let $B = (B_n, \delta_n)$ and $C = (C_n, d_n)$ be objects of $\mathrm{Kom}(\mathcal{A})$. We define something which we claim will be the product. The nth object of $B \times C$ will be the product $B_n \times C_n$. We have projections $\pi_{B_n} : B_n \times C_n \to B_n$ and $\pi_{C_n} : B_n \times C_n \to C_n$. By the universal property of the product, there is a unique morphism $\delta_n \times d_n$ (in $\mathcal{A}$) making the following diagram commute.

$$\begin{tikzcd}
 & B_n \times C_n \arrow[dl, "\delta_n \pi_{B_n}"'] \arrow[d, dotted, "\delta_n \times d_n"] \arrow[dr, "d_n \pi_{C_n}"] & \\
B_{n-1} & B_{n-1} \times C_{n-1} \arrow[l, "\pi_{B_{n-1}}"] \arrow[r, "\pi_{C_{n-1}}"'] & C_{n-1}
\end{tikzcd}$$

Then we claim that $(B_n \times C_n, \delta_n \times d_n)$ is a categorical product $B \times C$. We have a projection $(B_n \times C_n, \delta_n \times d_n) \to (B_n, \delta_n)$ which is just (π_{B_n}). By construction of $\delta_n \times d_n$, the required square commutes to make this a chain map.

$$\begin{tikzcd}
B_n \times C_n \arrow[r, "\delta_n \times d_n"] \arrow[d, "\pi_{B_{n-1}}"] & B_{n-1} \times C_{n-1} \arrow[d, "\pi_{B_{n-1}}"] \\
B_n \arrow[r, "\delta_n"] & B_{n-1}
\end{tikzcd}$$

Similarly, (π_{C_n}) is the projection to C. Now we just need to show that the universal property holds. Let $X = (X_n, \partial_n) \in \mathrm{Kom}(\mathcal{A})$ with morphisms $f = (f_n) : X \to B$ and $g = (g_n) : X \to C$. By the universal property of $B_n \times C_n$, there is a unique map $h_n : X_n \to B_n \times C_n$ making the following diagram commute.

$$\begin{tikzcd}
 & X_n \arrow[dl, "f_n"'] \arrow[d, dotted, "h_n"] \arrow[dr, "g_n"] & \\
B_n & B_n \times C_n \arrow[l, "\pi_{B_n}"] \arrow[r, "\pi_{C_n}"'] & C_n
\end{tikzcd}$$

Then we claim that $h = (h_n)$ is the unique morphism in $\mathrm{Kom}(\mathcal{A})$ making this diagram commute.

$$\begin{tikzcd}
 & X \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\
B & B \times C \arrow[l, "\pi_B"] \arrow[r, "\pi_C"'] & C
\end{tikzcd}$$

It is clear that this diagram commutes if h is a morphism, we just need to check that h is a chain map.

$$\begin{tikzcd}
X_n \arrow[r, "\partial_n"] \arrow[d, "h_n"] \arrow[dd, bend right=60, "f_n"'] & X_{n-1} \arrow[d, "h_{n-1}"] \arrow[dd, bend left=60, "f_{n-1}"] \\
B_n \times C_n \arrow[r, "\delta_n \times d_n"] \arrow[d, "\pi_{B_n}"] & B_{n-1} \times C_{n-1} \arrow[d, "\pi_{B_{n-1}}"] \\
B_n \arrow[r, "\delta_n"] & B_{n-1}
\end{tikzcd}
\qquad
\begin{tikzcd}
X_n \arrow[r, "\partial_n"] \arrow[d, "h_n"] \arrow[dd, bend right=60, "g_n"'] & X_{n-1} \arrow[d, "h_{n-1}"] \arrow[dd, bend left=60, "g_{n-1}"] \\
B_n \times C_n \arrow[r, "\delta_n \times d_n"] \arrow[d, "\pi_{C_n}"] & B_{n-1} \times C_{n-1} \arrow[d, "\pi_{C_{n-1}}"] \\
C_n \arrow[r, "d_n"] & C_{n-1}
\end{tikzcd}$$

We have $d_n g_n = g_{n-1}\partial_n$ and $\delta_n f_n = f_{n-1} d_n$. By the universal property of the product, there is a unique map making the following diagram commute.

$$\begin{tikzcd}
 & X_n \arrow[dl, "\partial_n f_n = f_{n-1} d_n = \pi_{B_{n-1}}(\delta_n \times d_n) h_n"'] \arrow[d, dotted] \arrow[dr, "d_n g_n = g_{n-1}\partial_n = \pi_{C_{n-1}} h_{n-1} \partial_n"] & \\
B_{n-1} & B_{n-1} \times C_{n-1} \arrow[l, "\pi_{B_{n-1}}"] \arrow[r, "\pi_{C_{n-1}}"'] & C_{n-1}
\end{tikzcd}$$

For the right triangle to commute, this unique map must be $h_{n-1}\partial_n$. For the left triangle to commute, the map must be $(\delta_n \times d_n)h_n$. Thus these maps are equal, so h is a chain map.

(7) Let $f : B \to C$ be a chain map and let $k_n : \ker f_n \to B_n$ be the kernel of f_n. We claim that the kernel of $f : B \to C$ is just the kernel of each $f_n : B \to C_n$. (Similar construction for cokernel.) By the universal property of $\ker f_{n-1}$, there is a unique map $\partial_n : \ker f_n \to \ker f_{n-1}$.

$$\begin{tikzcd}
 & B_{n-1} \arrow[dr, "f_{n-1}"] & \\
 & \ker f_{n-1} \arrow[u, "k_{n-1}"] \arrow[r, "0"] & C_{n-1} \\
\ker f_n \arrow[uur, bend left, "\delta_n k_n"] \arrow[ur, dotted, "\partial_n"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

This makes $(\ker f_n, \partial_n)$ a chain complex and $k = (k_n) : \ker f \to B$ a chain map.

$$\begin{tikzcd}
\ker f_n \arrow[r, "\partial_n"] \arrow[d, "d_n"] & \ker f_{n-1} \arrow[d, "k_{n-1}"] \\
B_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & B_{n-1} \arrow[d, "f_{n-1}"] \\
C_n \arrow[r, "d_n"] & C_{n-1}
\end{tikzcd}$$

We need to show that this chain complex satisfies the universal property of the kernel. Let $X = (X_n, \phi_n)$ and $g = (g_n) : X \to B$ so that $gf = 0$. Then $g_n f_n = 0$ for each n, so by the universal property of $\ker f_n$, there is a unique map $h_n : X_n \to \ker f_n$ in a suitable diagram. We claim $h = (h_n) : X \to K$ is a chain map.

$$\begin{tikzcd}
X_n \arrow[r, "\phi_n"] \arrow[d, "h_n"] \arrow[dd, bend right=60, "g_n"'] & X_{n-1} \arrow[d, "h_{n-1}"] \arrow[dd, bend left=60, "g_{n-1}"] \\
\ker f_n \arrow[r, "\partial_n"] \arrow[d, "k_n"] & \ker f_{n-1} \arrow[d, "k_{n-1}"] \\
B_n \arrow[r, "\delta_n"] & B_{n-1}
\end{tikzcd}$$

We know that everything except perhaps the top square commutes in this.

$$k_{n-1}h_{n-1}\phi_n = g_n\phi_n = \delta_n g_n = \delta_n k_n h_n = k_{n-1}\partial_n h_n$$

Then since k_{n-1} is a monomorphism, this implies $h_{n-1}\phi_n = \partial_n h_n$. Thus h is a chain map. We skip construction of the cokernel as it is just dual to this.

(8). First, note that $f : B \to C$ is monic if and only if each $f_n : B \to C_n$ is monic. Similarly, f is epi if and only if each f_n is epi. Then this should follow from the construction of the kernel as the kernel at each step and same for cokernel. $\square$

3 Homology Functors

As noted already, the study of an abstract abelian category is highly motivated by the category of R-modules. There, the reader has probably seen that we have functors H_n from chain complexes of R-modules back to R-modules. These functors are useful in many ways, such as giving algebraic invariants like singular homology in algebraic topology. If possible, we would like to generalize these functors to an arbitrary abelian category.

We wish to construct homology functors $H_n : \mathrm{Kom}(\mathcal{A}) \to \mathcal{A}$. In the category of R-modules, this is greatly simplified because we can conclude that $\operatorname{im} d_{n+1} \subset \ker d_n$ using only set-theoretic notions, but this becomes more complicated if $\mathcal{A}$ is a general abelian category. Here is our plan.

1. Define a canonical map $\operatorname{im} d_{n+1} \to \ker d_n$, and then define $H_n(C)$ to be the cokernel of the this map. (The object associated with the cokernel, not the morphism.)

2. Given a chain map $f : B \to C$, construct a "canonical" morphism $H_n(B) \to H_n(C)$.

3. Show that H_n is an additive covariant functor.

3.1 Definition of H_n on Complexes

Let $C = (C_n, d_n)$ be a chain complex in an abelian category $\mathcal{A}$. Let $k : \ker d_n \to C_n$ be the kernel of d_n. Let $q : C_n \to \operatorname{coker} d_{n+1}$ be the cokernel of d_{n+1}. Let $i : \operatorname{im} d_{n+1} \to C_n$ be the kernel of q, which is also the image of d_{n+1}. Let $\widetilde{d}_n : C_{n+1} \to \operatorname{im} d_{n+1}$ be the canonical epimorphism (see Proposition 1.5).

$$
\begin{tikzcd}
& & \operatorname{coker} d_{n+1} & & \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\widetilde{d}_{n+1}"] & C_n \arrow[u, "q"] \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots \\
& \operatorname{im} d_{n+1} \arrow[ur, "i"] & \ker d_n \arrow[u, "k"] & &
\end{tikzcd}
$$

Since C is a chain complex, $d_n d_{n+1} = 0$. By the proposition, $i\widetilde{d}_{n+1} = d_{n+1}$, so $d_n i \widetilde{d}_{n+1} = 0$. Since $\widetilde{d}_{n+1}$ is an epimorphism, this implies that $d_n i = 0$. Then by the universal property of k being the kernel of d_n, there is a unique map $\phi : \operatorname{im} d_{n+1} \to \ker d_n$ fitting in the following commutative diagram.

$$
\begin{tikzcd}
& C_n \arrow[dr, "d_n"] & \\
& \ker d_n \arrow[u, "k"] \arrow[r, "0"] & C_{n-1} \\
\operatorname{im} d_{n+1} \arrow[uur, bend left, "i"] \arrow[ur, dotted, "\phi"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}
$$

This map ϕ is what we call the canonical map $\operatorname{im} d_{n+1} \to \ker d_n$. As we said, we then define $H_n(C)$ to be the cokernel of ϕ.

Example: We want to confirm that this coincides with the usual homology functor when $\mathcal{A}$ is R-modules. In this concrete setting, $k : \ker d_n \to C_n$ is the inclusion map, and $\widetilde{d}_{n+1} = d_{n+1}$, and ϕ is also the inclusion $\operatorname{im} d_{n+1} \hookrightarrow \ker d_n$. In the category of R-modules, the cokernel of a map is the target mod the image. Since ϕ is injective, $\operatorname{im} \phi \cong \operatorname{im} d_{n+1}$, so $H_n(C) = \ker d_n / \operatorname{im} \phi = \ker d_n / \operatorname{im} d_{n+1}$, as it should be.

3.2 Definition of H_n on Chain Maps

Now we define how our alleged functor H_n acts on chain maps. Let $f = (f_n) : B \to C$ be a chain map. Let $\ell : \ker \delta_n \to B_n$ and $k : \ker d_n \to C_n$ be the respective kernels. Let $\widetilde{d}_{n+1}, \widetilde{\delta}_{n+1}$ be the respective canonical maps to the image, and let ϕ, ψ be the respective canonical maps used to construct $H_n(B), H_n(C)$. These all fit into the follow commutative diagram. (All squares commute.)

$$
\begin{tikzcd}
& \operatorname{im} \delta_{n+1} \arrow[r, "\psi"] & \ker \delta_n \arrow[r] \arrow[d, "\ell"] & \operatorname{coker} \psi = H_n(B) & \\
\cdots \arrow[r] & B_{n+1} \arrow[u, "\widetilde{\delta}_{n+1}"] \arrow[r, "\delta_{n+1}"] \arrow[d, "f_{n+1}"] & B_n \arrow[r, "\delta_n"] \arrow[d, "f_n"] & B_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "\widetilde{d}_{n+1}"] & C_n \arrow[r, "d_n"] & C_{n-1} \arrow[r] & \cdots \\
& \operatorname{im} d_{n+1} \arrow[r, "\phi"] & \ker d_n \arrow[r] \arrow[u, "k"] & \operatorname{coker} \phi = H_n(C) &
\end{tikzcd}
$$

We will construct maps α, β making the following diagram commute.

$$\begin{tikzcd}
\operatorname{im}\delta_{n+1} \arrow[r, "\psi"] \arrow[d, "\beta"] & \ker\delta_n \arrow[r] \arrow[d, "\alpha"] & \operatorname{coker}\psi = H_n(B) \\
\operatorname{im} d_{n+1} \arrow[r, "\phi"] & \ker d_n \arrow[r] & \operatorname{coker}\phi = H_n(C)
\end{tikzcd}$$

First, we construct α. Since $d_n f_n \ell = 0$, by the universal property of $\ker d_n$, there is a unique map $\alpha : \ker\delta_n \to \ker d_n$ so that $k\alpha = f_n\ell$.

$$\begin{tikzcd}
 & C_n \arrow[dr, "d_n"] & \\
 & \ker d_n \arrow[u, "k"] \arrow[r, "0"] & C_{n-1} \\
\ker\delta_n \arrow[uur, bend left, "f_n\ell"] \arrow[ur, dotted, "\alpha"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

Now we construct β. Observe that

$$0 = q d_{n+1} f_{n+1} = q f_n \delta_{n+1} = q f_n \ell \psi$$

By the universal property of $\operatorname{im} d_{n+1}$, there is a unique map $\beta : \operatorname{im}\delta_{n+1} \to \operatorname{im} d_{n+1}$ making the following diagram commute.

$$\begin{tikzcd}
 & C_n \arrow[dr, "q"] & \\
 & \ker d_n \arrow[u, "i"] \arrow[r, "0"] & \operatorname{coker} d_{n+1} \\
\ker\delta_n \arrow[uur, bend left, "f_n\ell\psi"] \arrow[ur, dotted, "\beta"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

We are now in the following situation.

$$\begin{tikzcd}
\operatorname{im}\delta_{n+1} \arrow[r, "\psi"] \arrow[d, "\beta"] & \ker\delta_n \arrow[r] \arrow[d, "\alpha"] & \operatorname{coker}\psi = H_n(B) \\
\operatorname{im} d_{n+1} \arrow[r, "\phi"] & \ker d_n \arrow[r] & \operatorname{coker}\phi = H_n(C)
\end{tikzcd}$$

The above diagram commutes because: $f_n\ell\psi = i\beta$ and $k\alpha = f_n\ell$, so $k\alpha\psi = f_n\ell\psi = i\beta = k\phi\beta$, this implies $\alpha\psi = \phi\beta$ since k is monic. Then $(\operatorname{coker}\phi) \circ \alpha\psi = (\operatorname{coker}\phi) \circ \phi\beta = 0$, and then by the universal property of the cokernel $H_n(B)$, there is a unique map $H_n(f) : H_n(B) \to H_n(C)$ making the following diagram commute.

$$\begin{tikzcd}
 & \ker\delta_n \arrow[d, "\operatorname{coker}\psi"] \arrow[ddr, bend left, "(\operatorname{coker}\phi)\circ\alpha"] & \\
\operatorname{im}\delta_{n+1} \arrow[ur, "\psi"] \arrow[r, "0"] \arrow[drr, bend right, "0"'] & H_n(B) \arrow[dr, dotted, "H_n(f)"] & \\
 & & H_n(C)
\end{tikzcd}$$

This is our definition of the morphism $H_n(f)$.

3.3 H_n is a Functor

Proposition 3.1. *H_n is an additive covariant functor.*

Proof. First, we show that $H_n(\mathrm{Id}_C) = \mathrm{Id}_{H_n(C)}$. This is relatively clear from the construction. The unique maps $\alpha : \operatorname{im} d_{n+1} \to \operatorname{im} d_{n+1}$ and $\beta : \ker d_n \to \ker d_n$ must be the respective identity maps, so $H_n(\mathrm{Id})$ must be the identity map on $\operatorname{coker} \phi$.

Let A, B, C chain complexes and $g : A \to B$ and $f : B \to C$ be chain maps. We need to verify that $H_n(g \circ f) = H_n(g) \circ H_n(f)$. Let $A = (A_n, \partial_n), B = (B_n, \delta_n), C = (C_n, d_n)$. We observe that all of the universal property constructions behave nicely with composition of chain maps, so by looking at the following diagram, it is not hard to convince ourselves that $H_n(g) \circ H_n(f) = H_n(g \circ f)$.

$$\begin{tikzcd}
\operatorname{im} \partial_{n+1} \arrow[r] \arrow[d] & \ker \partial_n \arrow[r] \arrow[d] & H_n(A) \arrow[d, "H_n(g)"] \\
\operatorname{im} \delta_{n+1} \arrow[r] \arrow[d] & \ker \delta_n \arrow[r] \arrow[d] & H_n(B) \arrow[d, "H_n(f)"] \\
\operatorname{im} d_{n+1} \arrow[r] & \ker d_n \arrow[r] & H_n(C)
\end{tikzcd}$$

Finally, we need to check that if $f_1, f_1 \in \operatorname{Hom}(C, D)$ then $H_n(f + f') = H_n(f) + H_n(f')$. As above, we are too lazy to work out the details, but all of the universal property constructions are compatible with the Hom-set addition operations, so

$$\begin{tikzcd}
\operatorname{im} \delta_{n+1} \arrow[r, "\psi+\psi'"] \arrow[d, "\beta+\beta'"] & \ker \delta_n \arrow[r] \arrow[d, "\alpha+\alpha'"] & H_n(B) \arrow[d, "H_n(f+f')=H_n(f)+H_n(f')"] \\
\operatorname{im} d_{n+1} \arrow[r, "\phi+\phi'"] & \ker d_n \arrow[r] & H_n(C)
\end{tikzcd}$$

$\square$

References

[1] Charles A. Weibel. *An Introduction to Homological Algebra.* Cambridge Studies in Advanced Mathematics, Vol. 38, Cambridge University Press, Cambridge, 1994.